

# Modular surfaces associated with toric K3 hypersurfaces

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## Abstract

We give detailed descriptions of the period maps of two 2-parameter families of anti-canonical hypersurfaces in toric 3-folds. One of them is related to a Hilbert modular surface, and the other is related to the product of modular curves.

## 1 Introduction

Mirror symmetry is a mysterious relationship between complex geometry and symplectic geometry motivated by string theory. Although mirror symmetry is most intensively studied for Calabi-Yau 3-folds, mirror symmetry for other classes of varieties, such as abelian varieties, Fano varieties, or varieties of general type, is also an interesting subject.

In this paper, we study the period maps of two 2-parameter families of K3 surfaces from the point of view of mirror symmetry. Similar analysis for the 1-parameter family of the quartic mirror K3 surfaces has been performed by Hartmann [Har13], based on earlier results by Nagura and Sugiyama [NS95] and Narumiya and Shiga [NS01].

The first family is mirror to anti-canonical hypersurfaces in the  $\mathbb{P}^1$ -bundle  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$  over  $\mathbb{P}^2$ . The parameter space of this family admits a natural compactification  $X_{\mathcal{F}(A_0)}$  called the *secondary stack*. The Picard lattice and the transcendental lattice of a very general member of this family are given by

$$M_0 = E_8 \perp E_8 \perp \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad T_0 = U \perp \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}. \quad (1.1)$$

The moduli space  $\mathcal{M}_0$  of  $M_0$ -polarized K3 surfaces is the symmetric Hilbert modular surface associated with  $\mathbb{Q}(\sqrt{5})$ .

**Theorem 1.1.** *The period map gives an isomorphism*

$$\tilde{\Pi} : \tilde{X}_{\mathcal{F}(A_0)} \xrightarrow{\sim} \mathcal{M}_{\Sigma_0}, \quad (1.2)$$

*of Deligne-Mumford stacks, where  $\tilde{X}_{\mathcal{F}(A_0)} \rightarrow X_{\mathcal{F}(A_0)}$  is a weighted blow-up of weight  $(1, 3)$  at one point followed by the root construction along the discriminant, and  $\mathcal{M}_{\Sigma_0}$  is a toroidal compactification of  $\mathcal{M}_0$  equipped with a natural orbifold structure.*

The *root construction* is the operation introduced in [AGV08, Cad07] which gives a generic stabilizer to a divisor. The fan  $\Sigma$  comes from the monodromy logarithm of the

period map around toric divisors of  $X_{\mathcal{F}(A_0)}$ . The inverse image in  $\mathcal{M}_\Sigma$  of the unique cusp in the Baily-Borel-Satake compactification  $\overline{\mathcal{M}}_0$  is the union of two toric divisors. The intersection point of these divisors is the maximally unipotent monodromy point. The monodromy logarithms  $N$  around these divisors satisfy  $N^2 \neq 0$  and  $N^3 = 0$ , so that one has type III degenerations there.

The second family is mirror to anti-canonical hypersurfaces in the toric weak Fano 3-fold of Picard number 2, obtained as a crepant resolution of a toric Fano 3-fold of Picard number 1 with an ordinary double point. The Picard lattice and the transcendental lattice of a very general member of this family are given by

$$M_1 = E_8 \perp E_8 \perp \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \quad T_0 = U \perp \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}. \quad (1.3)$$

The moduli space  $\mathcal{M}_1$  of  $M_1$ -polarized K3 surfaces is the quotient of the product  $\mathbb{H} \times \mathbb{H}$  of upper half planes under the action of  $\Gamma_0(3) \times \Gamma_0(3) \times C_2$ . Here each  $\Gamma_0(3)$  is a congruence subgroups of  $SL_2(\mathbb{Z})$  acting on each upper half plane, and  $C_2$  is the cyclic group of order 2 which permutes two upper half planes.

**Theorem 1.2.** *The period map gives an isomorphism*

$$\tilde{\Pi} : \tilde{X}_{\mathcal{F}(A_1)} \xrightarrow{\sim} \mathcal{M}_{\Sigma_1} \quad (1.4)$$

*of Deligne-Mumford stacks, where  $\tilde{X}_{\mathcal{F}(A_1)} \rightarrow X_{\mathcal{F}(A_1)}$  is the root construction along the discriminant, and  $\mathcal{M}_{\Sigma_1}$  is a toroidal compactification of  $\mathcal{M}_1$  equipped with a natural orbifold structure.*

The fan  $\Sigma_1$  comes from the monodromy logarithm of the period map around toric divisors of  $X_{\mathcal{F}(A_1)}$ . The cusp in the Baily-Borel-Satake compactification  $\overline{\mathcal{M}}_1$  is the union of two rational curves intersecting at one point, and the morphism  $\mathcal{M}_{\Sigma_1} \rightarrow \overline{\mathcal{M}}_1$  is the blow-up at this intersection point. There are two maximally unipotent monodromy point in  $\mathcal{M}_{\Sigma_1}$  on the exceptional curve, corresponding to two crepant resolutions of the toric Fano 3-fold. The monodromy logarithm  $N$  satisfies  $N \neq 0$  and  $N^2 = 0$  around these divisors, so that one has type II degenerations there.

The advantage of the secondary stack is that it comes with a natural family of toric hypersurfaces on it. This gives a family of degenerate K3 surfaces on the boundary of the toroidal compactification of the period domain. We expect that Theorems 1.1 and 1.2 admits an interpretation in terms of log period map of log K3 surfaces [Ols04, KU09].

This paper is organized as follows: In Section 2, we recall the notion of lattice-polarized K3 surfaces and compactifications of their moduli spaces. In Section 3, we recall a conjecture of Dolgachev [Dol96] on the relation between polar duality [Bat94] and mirror symmetry for lattice-polarized K3 surfaces. In Section 4, we recall the notion of the secondary stacks from [Laf03, Hac, DKK]. In Section 5, we describe the monodromy of the period map in terms of the autoequivalence of the derived category of coherent sheaves on the mirror manifold along the lines of [Iri11]. Theorem 1.1 is proved in Section 6, and Theorem 1.2 is proved in Section 7.

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## 2 Lattice polarized K3 surfaces

The *K3 lattice* is the even unimodular lattice  $L = E_8 \perp E_8 \perp U \perp U \perp U$  of rank 22 and signature  $(3, 19)$ . Here  $E_8$  is the negative-definite even unimodular lattice of type  $E_8$  and  $U$  is the indefinite even unimodular lattice of rank two. For a K3 surface  $Y$ , set

$$\Delta(Y) = \{\delta \in \text{Pic}(Y) \mid (\delta, \delta) = -2\}. \quad (2.1)$$

Let  $\mathcal{L}$  be a line bundle such that  $[\mathcal{L}] = \delta \in \Delta(Y)$ . Riemann-Roch theorem gives

$$h^0(\mathcal{L}) + h^0(\mathcal{L}^\vee) \geq 2 + \frac{1}{2}(\delta, \delta) = 1, \quad (2.2)$$

so that  $\mathcal{L}$  or  $\mathcal{L}^\vee$  has a non-trivial section and hence either  $\delta$  or  $-\delta$  is effective;

$$\Delta(Y) = \Delta(Y)^+ \amalg \Delta(Y)^-, \quad (2.3)$$

$$\Delta(Y)^+ = \{\delta \in \Delta(Y) \mid \delta \text{ is effective}\}, \quad (2.4)$$

$$\Delta(Y)^- = -\Delta(Y)^+. \quad (2.5)$$

The subgroup  $W(Y) \subset O(L)$  generated by reflections with respect to elements in  $\Delta(Y)$  acts properly discontinuously on the connected component

$$V^+ \subset V(Y) = \{x \in H^{1,1}(Y) \cap H^2(Y, \mathbb{R}) \mid (x, x) > 0\} \quad (2.6)$$

containing the Kähler class. The fundamental domain is given by

$$C(Y) = \{x \in V(Y)^+ \mid (x, \delta) \geq 0 \text{ for any } \delta \in \Delta(Y)^+\}, \quad (2.7)$$

and the Kähler cone is given (cf. e.g. [BHPVdV04, Corollary VIII.3.9]) by

$$C(Y)^+ = \{x \in V(Y)^+ \mid (x, \delta) > 0 \text{ for any } \delta \in \Delta(Y)^+\}. \quad (2.8)$$

Recall that

$$\text{Pic}(Y) = H^{1,1}(Y) \cap H^2(Y; \mathbb{Z}) \quad (2.9)$$

by the Lefschetz theorem. Set

$$\text{Pic}(Y)^+ = C(Y) \cap H^2(Y; \mathbb{Z}), \quad (2.10)$$

$$\text{Pic}(Y)^{++} = C(Y)^+ \cap H^2(Y; \mathbb{Z}). \quad (2.11)$$

Let  $M$  be an even non-degenerate lattice of signature  $(1, t)$  where  $0 \leq t \leq 19$ . Choose one of two connected components of

$$V(M) = \{x \in M_{\mathbb{R}} \mid (x, x) > 0\} \quad (2.12)$$

and call it  $V(M)^+$ . Choose a subset  $\Delta(M)^+$  of

$$\Delta(M) = \{\delta \in M \mid (\delta, \delta) = -2\} \quad (2.13)$$

such that

1.  $\Delta(M) = \Delta(M)^+ \amalg \Delta(M)^-$  where  $\Delta(M)^- = \{-\delta \mid \delta \in \Delta(M)^+\}$ , and
2.  $\Delta(M)^+$  is closed under addition (but not subtraction).

Define

$$C(M)^+ = \{h \in V(M)^+ \cap M \mid (h, \delta) > 0 \text{ for all } \delta \in \Delta(M)^+\}. \quad (2.14)$$

**Definition 2.1** (Dolgachev [Dol96]). An  $M$ -polarized K3 surface is a pair  $(Y, j)$  where  $Y$  is a K3 surface and  $j : M \hookrightarrow \text{Pic}(Y)$  is a primitive lattice embedding. An *isomorphism* of  $M$ -polarized K3 surfaces  $(Y, j)$  and  $(Y', j')$  is an isomorphism  $f : Y \rightarrow Y'$  of K3 surfaces such that  $j = f^* \circ j'$ . An  $M$ -polarized K3 surface is *ample* if

$$j(C(M)^+) \cap \text{Pic}(Y)^{++} \neq \emptyset. \quad (2.15)$$

Fix a primitive lattice embedding  $i_M : M \hookrightarrow L$  and let  $T$  be the orthogonal complement. The period domain

$$\mathcal{D} = \{[\Omega] \in \mathbb{P}(T_{\mathbb{C}}) \mid (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0\} \quad (2.16)$$

of  $M$ -polarized K3 surfaces can be identified with the symmetric homogeneous space  $O(2, 19 - t)/SO(2) \times O(19 - t)$  of oriented positive-definite 2-planes in  $T_{\mathbb{R}}$ . It consists of two onnected components  $\mathcal{D}^+$  and  $\mathcal{D}^-$ , each of which is isomorphic to a bounded Hermitian domain of type IV. Set

$$\Gamma(M) = \{\sigma \in O(L) \mid \sigma(m) = m \text{ for any } m \in M\} \quad (2.17)$$

and  $\Gamma$  be its image under the natural injective homomorphism

$$\Gamma(M) \hookrightarrow O(T). \quad (2.18)$$

Global Torelli Theorem and the surjectivity of the period map for K3 surfaces show that the moduli space of ample  $M$ -polarized K3 surfaces is bijective with  $\mathcal{D}^\circ/\Gamma$ , where

$$\mathcal{D}^\circ = \mathcal{D} \setminus \left( \bigcup_{\delta \in \Delta(T)} H_\delta \cap \mathcal{D} \right) \quad (2.19)$$

is the complement of reflection hyperplanes

$$H_\delta = \{z \in T_{\mathbb{C}} \mid (z, \delta) = 0\}. \quad (2.20)$$

The closure of the period domain in the *compact dual*

$$\check{\mathcal{D}} = \{[\Omega] \in \mathbb{P}(T_{\mathbb{C}}) \mid (\Omega, \Omega) = 0\} \quad (2.21)$$

of the period domain is denoted by  $\mathcal{D}^*$ . Its topological boundary is given by

$$\mathcal{D}^* \setminus \mathcal{D} = \bigcup_{I : \text{isotropic subspace of } M_{\mathbb{R}}} B(I), \quad (2.22)$$

where  $B(I)$  is defined by

$$B(I) = \mathbb{P}(I_{\mathbb{C}}) \setminus \left( \bigcup_{J \subsetneq I} \mathbb{P}(J_{\mathbb{C}}) \right). \quad (2.23)$$

Since the signature of  $M$  is  $(2, 19 - t)$ , one either has  $\text{rank } I = 1$  or  $2$ , so that  $\mathbb{P}(I_{\mathbb{C}}) \cap \mathcal{D}^*$  is one point or isomorphic to the upper half plane. The boundary component is *rational* if  $I$  is defined over  $\mathbb{Q}$ . The *Satake-Baily-Borel compactification* is defined by

$$\overline{\mathcal{D}}/\Gamma = \left( \mathcal{D} \cup \bigcup_{I : \text{rational}} \mathbb{P}(I_{\mathbb{C}}) \cap \mathcal{D}^* \right) / \Gamma. \quad (2.24)$$

Assume that one has  $T = U \perp N$  for a lattice  $N$ , and consider the neighborhood of the cusp corresponding to the isotropic subspace  $\mathbb{Z}e \subset T$ . Let  $\{e, f\}$  be a basis of  $U$  satisfying

$$(e, e) = (f, f) = 0, \quad (e, f) = 1, \quad (2.25)$$

and  $\Gamma_e$  be the stabilizer of  $e$  in  $O(T)$ . With an element  $v \in N$ , one can associate an isometry  $\varphi_{e,v} \in O(T)$  defined by

$$\varphi_{e,v}(x) = x - \left( \frac{1}{2}(v, v)(e, x) + (v, x) \right) e + (e, x)v. \quad (2.26)$$

One can easily see that

$$\varphi_{e,v} \circ \varphi_{e,w} = \varphi_{e,v+w}, \quad (2.27)$$

and

$$\varphi_{e,v}(e) = e, \quad \varphi_{e,v}(f) = -\frac{1}{2}(v, v)e + f + v, \quad \varphi_{e,v}(w) = -(v, w)e + w \quad (2.28)$$

for  $w \in N$ . It follows that  $\varphi_{e,\bullet}$  gives an embedding

$$\varphi_{e,\bullet} : N \hookrightarrow O(T) \quad (2.29)$$

of groups. Any element of  $\phi \in \Gamma_e$  can be written as  $\psi \circ \varphi_{e,v}$ , where  $v \in N$  is defined by  $\phi(f) \equiv f + v \pmod{\mathbb{Z}e}$  and  $\psi \in O(N)$ . This shows that

$$\Gamma_e = O(N) \ltimes N. \quad (2.30)$$

The period domain (2.16) can be realized as a tube domain

$$\{v = v_1 + \sqrt{-1}v_2 \in N_{\mathbb{C}} \mid v_i \in N_{\mathbb{R}}, (v_2, v_2) > 0\} \quad (2.31)$$

through the correspondence

$$\Omega = -\frac{1}{2}(v, v)e + f + v. \quad (2.32)$$

Under this correspondence, the action of  $\varphi_{e,u} \in N \subset \Gamma_e$  is given by translation  $v \mapsto v + u$ . Hence a neighborhood of the cusp of  $\mathcal{D}/\Gamma$  is locally isomorphic to

$$(N_{\mathbb{C}}/N)/O(N)^+ \cong N_{\mathbb{C}^\times}/O(N)^+. \quad (2.33)$$

Here  $O(N)^+$  is the subgroup of  $O(N)$  of index 2 preserving the connected component of  $\mathcal{D}^+$ . In other words,  $O(N)^+$  consists of elements whose norm and spinor norm have the same sign. Let  $\Sigma$  be a fan in  $N$  which is invariant under the action of  $O(N)^+$ . Then the toric variety  $X_\Sigma$  associated with  $\Sigma$  admits a natural action of  $O(N)^+$ . By replacing the neighborhood of the cusp with the neighborhood of the origin in the quotient  $X_\Sigma/O(N)^+$ , one obtains a toroidal partial compactification of  $\mathcal{D}/\Gamma$ .

### 3 Dolgachev conjecture

Let  $\mathbb{T} = (\mathbb{C}^\times)^n$  be an algebraic torus and  $\mathbf{M} = \text{Hom}(\mathbb{T}, \mathbb{C}^\times)$  be the group of characters. A convex lattice polytope  $\Delta \subset \mathbf{M}_\mathbb{R} = \mathbf{M} \otimes \mathbb{R}$  defines a projective toric variety  $X = \text{Proj } S_\Delta$  where  $S_\Delta$  is the monoid ring of the submonoid of  $\mathbf{M} \oplus \mathbb{N}$  consisting of lattice points of the cone over  $\Delta \times \{1\} \subset \mathbf{M}_\mathbb{R} \oplus \mathbb{R}$ . The *polar polytope* of  $\Delta$  is defined by

$$\check{\Delta} = \{v \in \check{\mathbf{M}} \mid \langle v, m \rangle \geq -1 \text{ for any } m \in \Delta\} \quad (3.1)$$

where  $\check{\mathbf{M}} = \text{Hom}(\mathbf{M}, \mathbb{Z})$  is the dual lattice of  $\mathbf{M}$ . The polytope  $\Delta$  is said to be *reflexive* if the polar dual polytope  $\check{\Delta}$  is a lattice polytope, and the origin is the unique interior lattice point of  $\Delta$ . The projective toric variety associated with  $\check{\Delta}$  will be denoted by  $\check{X} = \text{Proj } S_{\check{\Delta}}$ . The families  $|-K_X|$  and  $|-K_{\check{X}}|$  of anti-canonical hypersurfaces are called a *Batyrev mirror pair* [Bat94].

Assume  $n = 3$  and take very general members  $Y$  and  $\check{Y}$  of  $|-K_X|$  and  $|-K_{\check{X}}|$ . Define  $M_\Delta$  as the primitive sublattice of  $H^2(Y; \mathbb{Z})$  generated by the image of  $\iota^* : H^2(X; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z})$ , and similarly for  $M_{\check{\Delta}} \subset H^2(\check{Y}; \mathbb{Z})$ .

For a vector  $e$  in a lattice  $S$ , the positive integer  $\text{div } e$  is defined as the greatest common divisor of  $(e, f) \in \mathbb{Z}$  for all  $f \in S$ . A primitive isotropic vector  $e$  is called *m-admissible* if  $\text{div } e = m$  and there exists another primitive isotropic vector  $f$  such that  $(e, f) = m$  and  $\text{div } f = m$ .

Assume that  $M_\Delta^\perp \subset H^2(Y; \mathbb{Z})$  has an  $m$ -admissible vector  $e$ . Then one has  $M_\Delta^\perp = U(m) \perp \check{M}_\Delta$  where  $U(m)$  is the lattice generated by  $e$  and  $f$ , and  $\check{M}_\Delta$  is the orthogonal complement.

**Conjecture 3.1** (Dolgachev [Dol96, Conjecture (8.6)]).

1. The lattice  $M_\Delta^\perp$  contains a 1-admissible isotropic vector.
2. There exists a primitive embedding  $M_{\check{\Delta}} \subset \check{M}_\Delta$ .
3. The equality  $M_{\check{\Delta}} = \check{M}_\Delta$  holds if and only if  $M_\Delta \cong \text{Pic } Y$ .

### 4 Secondary stack

Let  $\Delta$  be a reflexive polytope in  $\mathbf{M}_\mathbb{R}$  and  $A = \{v_0 = 0, v_1, \dots, v_{n+r}\}$  be the set of lattice points of  $\Delta$ . It gives the *fan sequence*

$$0 \rightarrow \mathbb{L} \rightarrow \mathbb{Z}^{n+r} \xrightarrow{\beta} \check{\mathbf{N}} \rightarrow 0, \quad (4.1)$$

where  $\check{\mathbf{N}} = \mathbf{M}$  and  $\mathbb{L}$  is the kernel of the homomorphism  $P: \mathbb{Z}^{n+r} \rightarrow \check{\mathbf{N}}$  sending the  $i$ -th coordinate vector  $e_i$  to  $v_i$  for  $i = 1, \dots, n+r$ . We write a basis of  $\mathbb{L}$  as  $\{c^{(p)}\}_{p=1}^r$  where  $c^{(p)} = (c_1^{(p)}, \dots, c_{n+r}^{(p)})$ . By setting  $\tilde{v}_i = (v_i, 1) \in \check{\mathbf{N}} \oplus \mathbb{Z}$  for  $i = 0, \dots, n+r$ , one obtains a sequence

$$0 \rightarrow \mathbb{L} \rightarrow \mathbb{Z}^A \xrightarrow{\tilde{\beta}} \check{\mathbf{N}} \oplus \mathbb{Z} \rightarrow 0, \quad (4.2)$$

where the map  $\tilde{P}: \mathbb{Z}^A \cong \mathbb{Z}^{n+r+1} \rightarrow \check{\mathbf{N}} \oplus \mathbb{Z}$  sends  $e_i$  to  $\tilde{v}_i$  for  $i = 0, \dots, n+r$ , and an element  $c^{(p)} \in \mathbb{L}$  is mapped to

$$\tilde{c}^{(p)} = (-c_1^{(p)} - \dots - c_{n+r}^{(p)}, c_1^{(p)}, \dots, c_{n+r}^{(p)}) \in \mathbb{Z}^{n+r+1}.$$

The sequence

$$0 \rightarrow \check{\mathbf{M}} \xrightarrow{\beta^\vee} \mathbb{Z}^{n+r} \rightarrow \mathbb{L}^\vee \rightarrow 0 \quad (4.3)$$

dual to (4.1) is called the *divisor sequence*.

Given a polyhedral subdivision  $\underline{\Delta} = \{\Delta_1, \dots, \Delta_k\}$  of  $\Delta$ , one sets

$$C(\underline{\Delta}) = \{\psi \in \mathbb{R}^A \mid g_\psi \text{ is affine linear over each polytope in } \underline{\Delta}\}, \quad (4.4)$$

where  $g_\psi: \Delta \rightarrow \mathbb{R}$  is the convex piecewise linear function associated with  $\psi: A \rightarrow \mathbb{R}$ . The cone  $C(\underline{\Delta})$  is invariant under the additive action of the space  $\text{Aff}(\mathbf{M}_{\mathbb{R}})$  of affine linear functions on  $\mathbf{M}_{\mathbb{R}}$ . The quotient cones  $C(\underline{\Delta})/\text{Aff}(\mathbf{M}_{\mathbb{R}})$  constitute a complete fan  $\mathcal{F}(A)$  in  $\mathbb{R}^A/\text{Aff}(\mathbf{M}_{\mathbb{R}})$  called the *secondary fan* [GKZ94]. Maximal cones of the secondary fan  $\mathcal{F}(A)$  correspond to coherent triangulations of the polytope  $\Delta$ . A *circuit* is an affinely dependent subset any of whose proper subset is affinely independent [GKZ94, 7.1.B]. Adjacencies of triangulations come from modifications along circuits [GKZ94, Theorem 7.2.10].

The elements  $\{\tilde{c}^{(p)}\}_{p=1}^r$  generate one-dimensional cones of the secondary fan  $\mathcal{F}(A)$ , which gives a structure of a stacky fan on  $\mathcal{F}(A)$ . The toric stack  $X_{\mathcal{F}(A)}$  associated with the resulting stacky fan will be called the *secondary stack* [DKK]. The dense torus of the secondary stack can naturally be identified with  $\mathbb{L}_{\mathbb{C}^\times}$ . The coarse moduli space of  $X_{\mathcal{F}(A)}$  is the *Chow quotient* of  $\mathbb{P}(\mathbb{C}^A)$  by  $\mathbb{T}$  [KSZ91, KSZ92].

For a face  $\Delta'$  of  $\underline{\Delta}$  (i.e., a face of some  $\Delta_i$  in  $\underline{\Delta}$ ), set

$$C(\underline{\Delta}, \Delta') = \{\psi \in C(\underline{\Delta}) \mid g_\psi \text{ attains its minimum on } \Delta'\}.$$

The cone  $C(\underline{\Delta}, \Delta')$  is invariant under the action of constant functions in  $\text{Aff}(\mathbf{M}_{\mathbb{R}})$ . The abelian group  $\mathbb{Z}^A/\mathbb{Z}$  can naturally be identified with  $\mathbf{N}_H = \text{Hom}(\mathbb{C}^\times, H)$ . The cones  $C(\underline{\Delta}, \Delta')/\mathbb{R}$  constitutes a complete fan  $\tilde{\mathcal{F}}(A)$  in  $\mathbb{R}^A/\mathbb{R}$  called the *Lafforgue fan* [Laf03, Hac]. The toric stack associated with Lafforgue fan equipped with a stacky structure is called the *Lafforgue stack* [DKK]. The natural homomorphism  $\mathbb{Z}^A/\mathbb{Z} \cong \mathbf{N}_H \rightarrow \mathbb{Z}^A/\text{Aff}_{\mathbb{Z}}(\mathbf{M}) \cong \mathbf{N}$  defines a morphism  $\tilde{\mathcal{F}}(A) \rightarrow \mathcal{F}(A)$  of fans, which induces a torus-equivariant morphism  $\varphi_X: X_{\tilde{\mathcal{F}}(A)} \rightarrow X_{\mathcal{F}(A)}$  of toric stacks.

The Laurent polynomial

$$W = \sum_{i=0}^{n+r} a_i x_1^{v_{i1}} \cdots x_n^{v_{in}} \quad (4.5)$$

gives a section of  $(\varphi_X)_* \left( \mathcal{O}_{X(\tilde{F}(A))}(1) \right)$ , where  $\mathcal{O}_{X(\tilde{F}(A))}(1)$  is the line bundle which restricts to the anti-canonical bundle  $\mathcal{O}(-K_X)$  on a general fiber. The zero of  $W$  gives a family  $\varphi_Y: \mathfrak{Y}_A \rightarrow X_{\mathcal{F}(A)}$  of hypersurfaces.

The *discriminantal variety*  $\nabla_A \subset \mathbb{C}^A$  is the closure of all  $a \in \mathbb{C}^A$  such that there exists  $x \in (\mathbb{C}^\times)^n$  satisfying

$$W = \frac{\partial W}{\partial x_1} = \dots = \frac{\partial W}{\partial x_n} = 0. \quad (4.6)$$

The *A-discriminant* is the irreducible polynomial  $\Delta_A \in \mathbb{Z}[a_0, \dots, a_{n+r}]$  vanishing on  $\nabla_A$ . The Newton polytope of  $\Delta_A$  is the *secondary polytope* of  $A$ . The normal fan to the secondary polytope is the secondary fan  $\mathcal{F}(\Sigma)$ . Since  $\nabla_A \cap (\mathbb{C}^\times)^A$  is invariant under the action of  $\mathbb{T} \times \mathbb{C}^\times$ , it descends to a hypersurface  $\overline{\nabla}_A \subset \mathbb{L}_{\mathbb{C}^\times}$  called the *reduced A-discriminantal variety*. The *Horn-Kapranov uniformization* [Hor89, Kap91] is the rational map  $h: \mathbb{P}^r \rightarrow \mathbb{L}_{\mathbb{C}^\times}$ ,  $\lambda = [\lambda_1 : \dots : \lambda_{r+1}] \mapsto (\Phi_1(\lambda_1), \dots, \Phi_r(\lambda))$  where

$$\Phi_q(\lambda) = \prod_{j=1}^n \left( \sum_{p=1}^r c_j^{(p)} \lambda_p \right)^{c_j^{(q)}}. \quad (4.7)$$

The image of  $h$  is the reduced *A-discriminantal variety*. The restriction of the family  $\varphi_Y$  to the complement  $X_{\mathcal{F}(A)}^{\text{reg}} = \mathbb{L}_{\mathbb{C}^\times} \setminus \nabla_A$  of the discriminantal variety will be denoted by

$$\varphi_Y^{\text{reg}}: \mathfrak{Y}_A^{\text{reg}} \rightarrow X_{\mathcal{F}(A)}^{\text{reg}}. \quad (4.8)$$

## 5 Mirror symmetry and monodormy

Hodge theory gives an integral variation  $(H_{B,\mathbb{Z}}^{\text{vc}}, \nabla^B, \mathcal{F}_B^\bullet, Q_B)$  of polarized pure Hodge structures [Iri11, Definitions 6.5 and 6.7], where

- $H_{B,\mathbb{Z}}^{\text{vc}}$  is the local system on  $X_{\mathcal{F}(A)}^{\text{reg}}$  whose fiber over  $[a] \in X_{\mathcal{F}(A)}^{\text{reg}}$  is the sublattice of  $H^{n-1}(Y_a, \mathbb{Z})$  generated by vanishing cycles,
- $\nabla^B$  is the Gauss-Manin connection on  $H_{B,\mathbb{Z}}^{\text{vc}} \otimes \mathcal{O}_{X_{\mathcal{F}(A)}^{\text{reg}}}$ ,
- $\mathcal{F}_B^\bullet$  is the Hodge filtration, and
- $Q_B$  is the polarization.

On the mirror side, let

$$H_{\text{amb}}^\bullet(\check{Y}; \mathbb{C}) = \text{Im}(\iota^*: H^\bullet(\check{X}; \mathbb{C}) \rightarrow H^\bullet(\check{Y}; \mathbb{C})) \quad (5.1)$$

be the subspace of  $H^\bullet(\check{Y}; \mathbb{C})$  coming from the cohomology classes of the ambient toric variety, and set

$$U = \{ \sigma = \beta + \sqrt{-1}\omega \in H_{\text{amb}}^2(\check{Y}; \mathbb{C}) \mid \langle \omega, d \rangle \gg 0 \text{ for any non-zero } d \in \text{NE}(\check{Y}) \}, \quad (5.2)$$

where  $\text{NE}(\check{Y})$  is the semigroup of effective curves. Let  $\{p_i\}_{i=1}^r$  be an integral basis of the nef cone of  $\check{Y}$ , and  $(\sigma^i)_{i=1}^r$  be the dual coordinate on  $H_{\text{amb}}^2(\check{Y}; \mathbb{C})$ . The *ambient A-model VHS*  $(H_{A,\mathbb{Z}}^{\text{amb}}, \nabla^A, \mathcal{F}_A^\bullet, Q_A)$  consists of a local system  $H_{A,\mathbb{Z}}^{\text{amb}}$  on  $U$ , the Dubrovin connection

$$\nabla^A = d + \sum_{i=1}^r (p_i \circ \sigma) d\sigma^i: \mathcal{H}_A \rightarrow \mathcal{H}_A \otimes \Omega_U^1, \quad (5.3)$$



on  $\mathcal{H}_A = H_{\text{amb}}^\bullet(\check{Y}; \mathbb{C}) \otimes \mathcal{O}_U$ , the Hodge filtration

$$\mathcal{F}_A^p = H_{\text{amb}}^{4-2p}(\check{Y}; \mathbb{C}) \otimes \mathcal{O}_U,$$

and the Poincaré pairing

$$Q_A : \mathcal{H}_A \otimes \mathcal{H}_A \rightarrow \mathcal{O}_U.$$

See [Iri11, Definition 6.2] for details. The fiber of the local system  $H_{A, \mathbb{Z}}^{\text{amb}}$  is isomorphic to the subgroup  $\mathcal{N}_{\text{amb}}(\check{Y})$  of the numerical Grothendieck group  $\mathcal{N}(\check{Y})$  generated by classes pulled-back from  $\mathcal{N}(\check{X})$ . The numerical Grothendieck group  $\mathcal{N}(\check{Y})$  is the quotient of the Grothendieck group  $K(\check{Y})$  by the radical of the Euler form

$$\chi(\mathcal{E}, \mathcal{F}) := \sum_{i=0}^2 (-1)^i \dim \text{Ext}^i(\mathcal{E}, \mathcal{F}). \quad (5.4)$$

Riemann-Roch theorem shows that  $\mathcal{N}(\check{Y})$  is isomorphic to the image of  $K(\check{Y})$  by the map

$$v : K(\check{Y}) \rightarrow H^*(\check{Y}, \mathbb{Q}), \quad \mathcal{E} \mapsto \text{ch}(\mathcal{E}) \cup \widehat{\Gamma}_{\check{Y}}, \quad (5.5)$$

where the  $\widehat{\Gamma}$ -class is a square root of the Todd class [Iri09]. The polarization  $Q_A$  is mapped to minus the Euler form under this isomorphism.

Let  $u_i \in H^2(X; \mathbb{Z})$  be the Poincaré dual of the toric divisor corresponding to the one-dimensional cone  $\mathbb{R} \cdot v_i \in \Sigma$  and  $v = u_1 + \dots + u_m$  be the anticanonical class. Givental's *I-function* is defined as the series

$$I_{X,Y}(q, z) = e^{p \log q / z} \sum_{d \in \text{NE}(X)} q^d \frac{\prod_{k=-\infty}^{\langle d, v \rangle} (v + kz) \prod_{j=1}^m \prod_{k=-\infty}^0 (u_j + kz)}{\prod_{k=-\infty}^0 (v + kz) \prod_{j=1}^m \prod_{k=-\infty}^{\langle d, u_j \rangle} (u_j + kz)}, \quad (5.6)$$

which is a map from  $\check{U}$  to the classical cohomology ring  $H^\bullet(X; \mathbb{C}[z^{-1}])$ . When  $Y$  is a K3 surface, Givental's *J-function* is given by

$$J_Y(\tau, z) = \exp(\tau/z). \quad (5.7)$$

If we write

$$I_{X,Y}(q, z) = F(q) + \frac{G(q)}{z} + \frac{H(q)}{z^2} + O(z^{-3}), \quad (5.8)$$

then Givental's mirror theorem [Giv96, Giv98, CG07] states that

$$\text{Euler}(\omega_X^{-1}) \cup I_{X,Y}(q, z) = F(q) \cdot \iota_* J_Y(\varsigma(q), z) \quad (5.9)$$

where  $\text{Euler}(\omega_X^{-1}) \in H^2(X; \mathbb{Z})$  is the Euler class of the anticanonical bundle of  $X$ , and the *mirror map*  $\varsigma(q) : \check{U} \rightarrow H_{\text{amb}}^2(Y; \mathbb{C})$  is defined by

$$\varsigma(q) = \iota^* \left( \frac{G(q)}{F(q)} \right). \quad (5.10)$$

The relation between  $\tau = \varsigma(q)$  and  $\sigma = \beta + \sqrt{-1}\omega$  is given by  $\tau = 2\pi\sqrt{-1}\sigma$ , so that  $\Im(\sigma) \gg 0$  corresponds to  $\exp(\tau) \sim 0$ . The functions  $F(q)$ ,  $G(q)$  and  $H(q)$  satisfy the Gelfand–Kapranov–Zelevinsky hypergeometric differential equations, and give periods for the B-model VHS.

**Theorem 5.1** (Iritani [Iri11, Theorem 6.9]). *There is an isomorphism*

$$\mathrm{Mir}_Y : \varsigma^*(H_{A,\mathbb{Z}}^{\mathrm{amb}}, \nabla^A, \mathcal{F}_A^\bullet, Q_A) \xrightarrow{\sim} (H_{B,\mathbb{Z}}^{\mathrm{vc}}, \nabla^B, \mathcal{F}_B^\bullet, Q_B) \quad (5.11)$$

*of integral variations of pure and polarized Hodge structures.*

The main step in the proof of Theorem 5.1 is [Iri11, Theorem 5.7], which relates periods of A-model VHS and those of B-model VHS. In the proof of [Iri11, Theorem 5.7], Iritani shows that the monodromy of the B-model VHS along a small loop  $q_i \mapsto e^{2\pi\sqrt{-1}}q_i$  is mapped to the isometry

$$(-) \otimes \iota^*(\mathcal{L}_i^\vee) : \mathcal{N}(\check{Y}) \rightarrow \mathcal{N}(\check{Y}) \quad (5.12)$$

where  $\mathcal{L}_i$  is the line bundle on  $\check{X}$  with  $c_1(\mathcal{L}_i) = p_i$ . Note that this isometry comes from an autoequivalence of  $D^b \mathrm{coh} \check{Y}$ . The relation between monodromy of the periods and autoequivalences of the derived category of coherent sheaves on the mirror manifold goes back to [Kon98, Hor05].

When  $\check{Y}$  is a K3 surface, then the numerical Grothendieck group  $\mathcal{N}(\check{Y})$  is the direct sum  $\mathbb{Z}[\mathcal{O}_{\check{Y}}] \oplus \mathrm{Pic}(\check{Y}) \oplus \mathbb{Z}[\mathcal{O}_p]$  of the Picard group  $\mathrm{Pic}(\check{Y})$  and the free module generated by the classes of the structure sheaf and a skyscraper sheaf. The embedding of the Picard group to the numerical Grothendieck group is given by  $[\mathcal{O}_{\check{Y}}(D)] \mapsto [\mathcal{O}_D]$ . The lattice structure is given by

$$(\mathcal{O}_{\check{Y}}, \mathcal{O}_{\check{Y}}) = -2, (\mathcal{O}_{\check{Y}}, \mathcal{O}_D) = -\chi(\mathcal{O}_D), (\mathcal{O}_{\check{Y}}, \mathcal{O}_p) = -1, \quad (5.13)$$

$$(\mathcal{O}_D, \mathcal{O}_E) = D \cdot E, (\mathcal{O}_D, \mathcal{O}_p) = -\chi(\mathcal{O}_p, \mathcal{O}_p) = 0, \quad (5.14)$$

where  $\mathcal{O}_{\check{Y}}$  is the structure sheaf,  $\mathcal{O}_D$  is the structure sheaf of a divisor  $D$ , and  $\mathcal{O}_p$  is a skyscraper sheaf. If  $D$  is a smooth curve of genus  $g$ , then one has  $\chi(\mathcal{O}_D) := \dim H^0(\mathcal{O}_D) - \dim H^1(\mathcal{O}_D) = 1 - g$  and  $D \cdot D = 2g - 2$ . The action of  $(-) \otimes \mathcal{O}_{\check{Y}}(-D) : K(\check{Y}) \rightarrow K(\check{Y})$  is given by

$$[\mathcal{O}_{\check{Y}}] \mapsto [\mathcal{O}(-D)] = [\mathcal{O}_{\check{Y}}] - [\mathcal{O}_D], \quad (5.15)$$

$$[\mathcal{O}_E] \mapsto [\mathcal{O}_E(-D)] = [\mathcal{O}_E] - (D \cdot E)[\mathcal{O}_p], \quad (5.16)$$

$$[\mathcal{O}_p] \mapsto [\mathcal{O}_p]. \quad (5.17)$$

## 6 The family associated with $A_0$

### 6.1 The secondary fan

Let  $\Delta_0 = \mathrm{Conv}\{v_1, v_2, v_3, v_4, v_5\}$  be the reflexive polytope with 5 vertices in Figure 6.1;

$$\beta_0 = (v_1 \ v_2 \ v_3 \ v_4 \ v_5) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}. \quad (6.1)$$

The homomorphism  $\mathbb{Z}^{n+r} \rightarrow \mathbb{L}^\vee$  in the divisor sequence (4.3) is represented by the matrix

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 1 & 0 \\ -5 & 1 & 1 & 2 & 0 & 1 \end{pmatrix}, \quad (6.2)$$

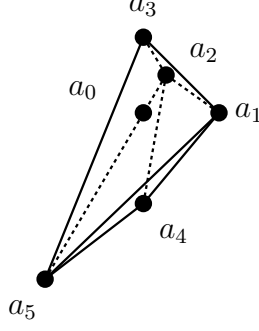


Figure 6.1: The polytope  $\Delta_0$

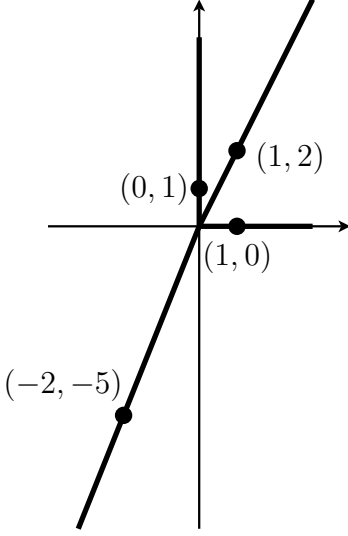


Figure 6.2: The secondary fan  $\mathcal{F}(A_0)$

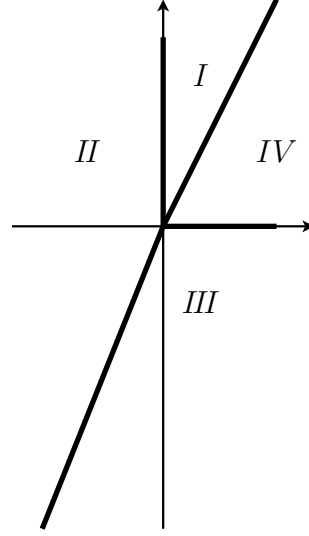


Figure 6.3: Maximal cones of  $\mathcal{F}(A_0)$

and the secondary fan is given in Figure 6.2. Figure 6.3 shows maximal cones of the secondary fan  $\mathcal{F}(A_0)$ . The cone  $I$  corresponds to the large complex structure limit point. The corresponding coherent triangulation is given by

$$\{a_0, a_1, a_2, a_3\} \cup \{a_0, a_2, a_3, a_5\} \cup \{a_0, a_1, a_3, a_5\} \cup \{a_0, a_1, a_2, a_4\} \cup \{a_0, a_2, a_4, a_5\} \cup \{a_0, a_1, a_4, a_5\}.$$

In the cone  $II$ , it is given by

$$\{a_1, a_2, a_3, a_4\} \cup \{a_2, a_3, a_4, a_5\} \cup \{a_1, a_3, a_4, a_5\},$$

which is the ‘vertical’ triangulation. In the cone  $III$ , it is given by

$$\{a_1, a_2, a_3, a_5\} \cup \{a_1, a_2, a_4, a_5\},$$

which is the ‘horizontal’ triangulation. In the cone  $IV$ , it is given by

$$\{a_0, a_1, a_2, a_3\} \cup \{a_0, a_1, a_3, a_5\} \cup \{a_0, a_1, a_2, a_5\} \cup \{a_0, a_2, a_3, a_5\} \cup \{a_1, a_2, a_4, a_5\}.$$

$I$  is obtained from  $II$  by the subdivision at  $a_0$ , i.e., the modification along the circuit  $\{a_0, a_3, a_4\}$ . The family over the corresponding divisor is the union of three rational surfaces intersecting along three rational curves. All these three curves pass through

two points, so that the dual graph of the intersection is the division of  $S^2$  into two triangles. *II* and *III* are related by the modification along the circuit  $\{a_1, a_2, a_3, a_4, a_5\}$ . The family parametrized by the corresponding divisor is a family of K3 surfaces. *III* and *IV* are related by the subdivision of  $\{a_1, a_2, a_3, a_5\}$ , i.e., the modification along the circuit  $\{a_0, a_1, a_2, a_3, a_5\}$ . Note that one has

$$\begin{aligned}\text{vol}\{a_1, a_2, a_3, a_5\} &= 5, \\ \text{vol}\{a_0, a_1, a_2, a_5\} &= 2.\end{aligned}$$

The corresponding family is the family of K3 surfaces associated with the polytope  $\{a_1, a_2, a_3, a_5\}$ . *IV* and *I* are related by the modification along the circuit  $\{a_0, a_1, a_2, a_4, a_5\}$ . The corresponding family is the union of four rational surfaces whose dual graph is a tetrahedron.

## 6.2 The period domain

The Picard lattice and the transcendental lattice of a very general member of the family  $\mathfrak{Y}_{A_0} \rightarrow X_{\mathcal{F}(A_0)}$  of K3 surfaces associated with  $A_0$  is given as follows:

**Theorem 6.1** ([Nag12, Theorem 3.1]). *The Picard lattice and the transcendental lattice of a very general member of the family  $\mathfrak{Y}_{A_0} \rightarrow X_{\mathcal{F}(A_0)}$  are given by*

$$M_0 = E_8 \perp E_8 \perp \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad T_0 = U \perp \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}. \quad (6.3)$$

The moduli space of  $M_0$ -polarized K3 surfaces can be identified with a Hilbert modular surface as follows. Let  $\mathcal{O}$  be the ring of integers of the real quadratic field  $\mathbb{Q}(\sqrt{5})$ . The Hilbert modular group  $PSL_2(\mathcal{O})$  acts on the product  $\mathbb{H} \times \mathbb{H}$  of the upper half planes by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (z_1, z_2) \mapsto \left( \frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_1 + \beta'}{\gamma' z_1 + \delta'} \right), \quad (6.4)$$

where  $(-)'$  is the conjugation in  $\mathbb{Q}(\sqrt{5})$ . One has

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = W U W^T, \quad W = \begin{pmatrix} 1 & 1 \\ -\varepsilon^{-1} & \varepsilon \end{pmatrix}, \quad \varepsilon = (1 + \sqrt{5})/2, \quad (6.5)$$

so that

$$\mathbb{H} \times \mathbb{H} \rightarrow \mathcal{D}^+, \quad (z_1, z_2) \mapsto (I_2 \oplus (W^T)^{-1}) \begin{pmatrix} z_1 z_2 \\ -1 \\ z_1 \\ z_2 \end{pmatrix} \quad (6.6)$$

gives a biholomorphic map. The orthogonal group  $PO^+(T_0)$  is generated by the Hilbert modular group  $PSL_2(\mathcal{O})$  and the permutation

$$\tau : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}, \quad (z_1, z_2) \mapsto (z_2, z_1) \quad (6.7)$$

under this identification. The symmetric Hilbert modular surface  $\mathbb{H} \times \mathbb{H} / \langle PSL_2(\mathcal{O}), \tau \rangle$  is studied in detail by Hirzebruch [Hir77] (cf. also [KKN89]). The graded ring  $\mathfrak{M} = \bigoplus_{n=0}^{\infty} \mathfrak{M}_n$  of symmetric Hilbert modular forms is generated by forms  $\mathfrak{A}$ ,  $\mathfrak{B}$ ,  $\mathfrak{C}$ ,  $\mathfrak{D}$  of weights 2, 6, 10, 15 with one relation of degree 30;

$$\mathfrak{M} = \mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}] / (144\mathfrak{D}^2 - \Delta(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})), \quad (6.8)$$

$$\Delta(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) = -1728\mathfrak{B}^5 + 720\mathfrak{A}\mathfrak{B}^3\mathfrak{C} - 80\mathfrak{A}^2\mathfrak{B}\mathfrak{C}^2 + 64\mathfrak{A}^3(5\mathfrak{B}^2 - \mathfrak{A}\mathfrak{C})^2 + \mathfrak{C}^3. \quad (6.9)$$

The stack  $\overline{\mathcal{M}} = \mathbb{P}\text{roj } \mathfrak{M} = [(\text{Spec } \mathfrak{M} \setminus \mathbf{0}) / \mathbb{C}^\times]$  is a hypersurface of degree 30 in the weighted projective space  $\mathbb{P}(2, 6, 10, 15)$ . It is obtained from the weighted projective plane  $\mathbb{P}(1, 3, 5) = \mathbb{P}\text{roj } \mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}]$  by the root construction along the divisor defined by  $\Delta$ . The cusp consists of one point  $[\mathfrak{A} : \mathfrak{B} : \mathfrak{C}] = [1 : 0 : 0]$ .

### 6.3 The period map

The space of Laurent polynomials is given by

$$\mathbb{C}^{A_0} = \left\{ W = a_0 + a_1x + a_2y + a_3z + \frac{a_4}{z} + \frac{a_5}{xyz^2} \right\}, \quad (6.10)$$

and the discriminant is given by

$$\begin{aligned} \Delta = & a_4^2a_0^6 + 4a_1a_2a_5a_0^5 - 12a_3a_4^3a_0^4 - 50a_1a_2a_3a_4a_5a_0^3 \\ & + 48a_3^2a_4^4a_0^2 + 1000a_1a_2a_3^2a_4^2a_5a_0 - 64a_3^3a_4^5 + 3125a_1^2a_2^2a_3^2a_5^2. \end{aligned} \quad (6.11)$$

The dense torus  $\mathbb{L}_{\mathbb{C}^\times}^\vee$  of the secondary stack  $X_{\mathcal{F}(A_0)}$  can be written as  $\text{Spec } \mathbb{C}[\lambda^{\pm 1}, \mu^{\pm 1}]$  where

$$\lambda = \frac{a_2a_4}{a_0^2}, \quad (6.12)$$

$$\mu = \frac{a_1a_2a_3^2a_5}{a_0^5}. \quad (6.13)$$

The period map

$$X_{\mathcal{F}(A_0)} \dashrightarrow \mathbb{P}(1, 3, 5) \quad (6.14)$$

for the family  $\mathcal{Y}_0$  is computed in [Nag12, Theorem 6.2] as

$$(\lambda, \mu) \mapsto \left[ 1 : \frac{25\mu}{2(\lambda - 1/4)^3} : -\frac{3125\mu^2}{(\lambda - 1/4)^5} \right]. \quad (6.15)$$

Since  $X_{\mathcal{F}(A_0)}$  is a weighted blow-up of  $\mathbb{P}(1, 2, 5)$ , the period map induces a rational map

$$\mathbb{P}(1, 2, 5) \dashrightarrow \mathbb{P}(1, 3, 5), \quad (6.16)$$

which is given by

$$[\nu : \lambda : \mu] \mapsto \left[ \lambda - \nu^2/4 : \frac{25}{2}\nu\mu : -3125\mu^2 \right]. \quad (6.17)$$

This map is not defined at  $[\nu : \lambda : \mu] = [1 : 1/4 : 0]$ . The weighted blow-up of weight  $(1, 3)$  along the ideal  $(\lambda - \nu^2/4, \mu)$  eliminates the indeterminacy, and the resulting morphism is a weighted blow-down of weight  $(1, 2)$  which contracts the strict transform of the divisor  $\{\mu = 0\} \subset \mathbb{P}(1, 2, 5)$  to the point  $[1 : 0 : 0] \in \mathbb{P}(1, 3, 5)$ . Since the secondary stack  $X_{\mathcal{F}(A_0)}$  is obtained from  $\mathbb{P}(1, 2, 5)$  by a weighted blow-up of weight  $(1, 2)$  at one point, the indeterminacy of the rational map  $\Pi: X_{\mathcal{F}(A_0)} \dashrightarrow \mathbb{P}(1, 3, 5)$  can also be eliminated by a weighted blow-up  $\tilde{X}_{\mathcal{F}(A_0)} \rightarrow X_{\mathcal{F}(A_0)}$  of weight  $(1, 3)$ , and the resulting morphism  $\tilde{\Pi}: \tilde{X}_{\mathcal{F}(A_0)} \rightarrow \mathbb{P}(1, 3, 5)$  is an iteration of weighted blow-ups of weight  $(1, 2)$ . A schematic picture of the blow-ups of the parameter space is shown in Figure 6.4.

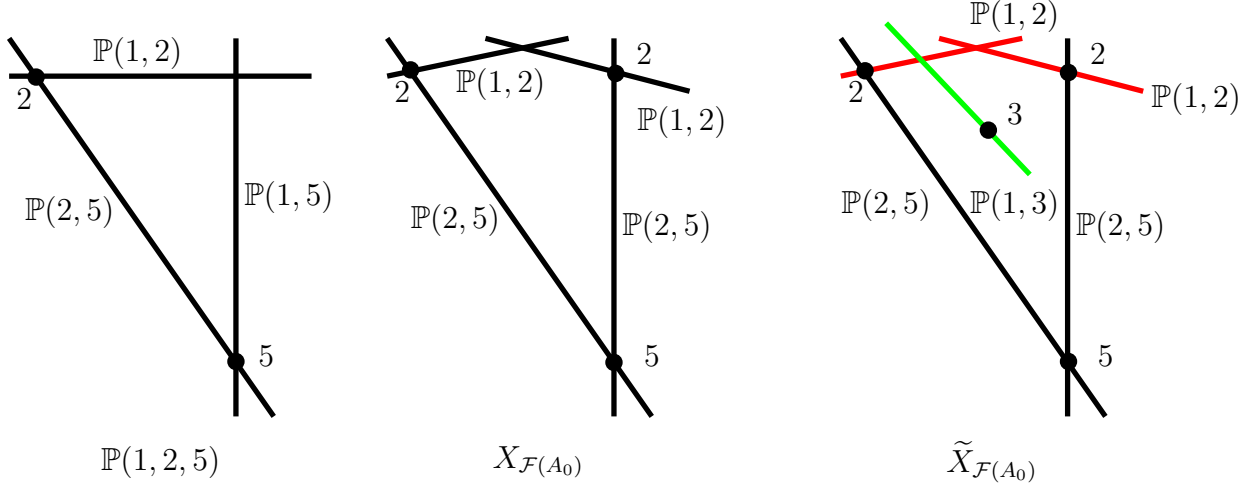


Figure 6.4: The blow-ups of parameter spaces

## 6.4 Mirror symmetry and monodromy

The polar dual polytope of  $\Delta_0$  is given by

$$\tilde{\Delta}_0 = \text{Conv} \{ (0, -1, 1), (4, -1, -1), (-1, -1, -1), (-1, -1, 1), ((-1, 4, -1), (-1, 0, 1)) \}.$$

The associated toric variety  $\check{X}$  is a toric Fano manifold, which is a  $\mathbb{P}^1$ -bundle  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$  over  $\mathbb{P}^2$ . The following theorem of Mořsezon shows that the Picard lattice of a very general member  $\check{Y}$  of the family  $\check{\mathfrak{Y}}$  is generated by the restrictions of the toric divisors of the ambient space:

**Theorem 6.2** ([Moř67, Theorem 7.5]). *Let  $V$  be a smooth projective 3-fold and  $\iota: E \hookrightarrow V$  be a very general hyperplane section. Then the map  $\iota^*: \text{Pic}(E) \rightarrow \text{Pic}(V)$  is surjective if and only if one of the following holds:*

1. The Betti numbers satisfy  $b_2(V) = b_2(E)$ .
2. The Hodge numbers satisfy  $h^{2,0}(E) > h^{2,0}(V)$ .

It follows that

$$M_{\Delta_0} = \text{Pic } Y_0 = E_8 \perp E_8 \perp \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad (6.18)$$

$$M_{\Delta_0}^\perp = U \perp \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad (6.19)$$

$$M_{\check{\Delta}_0} = \text{Pic } \check{Y}_0 = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad (6.20)$$

$$M_{\check{\Delta}_0}^\perp = U \perp E_8 \perp E_8 \perp \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad (6.21)$$

so that Conjecture 3.1 holds in this case.

The nef cone of  $\check{X}$  is generated by  $2D_1 + D_4$  and  $D_1$ , where  $D_1$  and  $D_4$  are toric divisors associated with one-dimensional cones generated by  $v_1$  and  $v_4$ . Let  $E_1$  and  $E_2$  be the restrictions of  $2D_1 + D_4$  and  $D_1$  to  $\check{Y}$  respectively. The corresponding coordinates  $(q_1, q_2)$  of  $H_{\text{amb}}^2(\check{Y}, \mathbb{C}^\times)$  near the large radius limit can be identified with the coordinates  $(\lambda, \mu)$  of the dense torus  $\mathbb{L}_{\mathbb{C}^\times}^\vee \subset X_{\mathcal{F}(A_0)}$  of the secondary stack by

$$q_1 = \lambda, \quad (6.22)$$

$$q_2 = \frac{\mu}{\lambda^2}. \quad (6.23)$$

The algebraic lattice of  $\check{Y}$  can be written as  $\mathcal{N}(\check{Y}) \cong U \perp N$ , where the hyperbolic plane  $U = \mathbb{Z}e \oplus \mathbb{Z}f$  is generated by  $e = [\mathcal{O}_p]$  and  $f = -[\mathcal{O}_{\check{Y}}] - [\mathcal{O}_{E_1}] + 3[\mathcal{O}_{E_2}]$ . The orthogonal complement

$$N \cong \begin{pmatrix} 10 & 5 \\ 5 & 2 \end{pmatrix} \quad (6.24)$$

is generated by  $\{e_1 = [\mathcal{O}_{E_1}], e_2 = [\mathcal{O}_{E_2}]\}$  and isometric to the Néron-Severi lattice  $\text{NS}(\check{Y})$ . The orthogonal group of  $N$  is generated by two elements;

$$O^+(N) = \langle g_1, g_2 \rangle, \quad g_1 = \begin{pmatrix} 4 & 1 \\ -5 & -1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}. \quad (6.25)$$

The monodromies  $T_1$  and  $T_2$  for  $(q_1, q_2) \mapsto (e^{2\pi\sqrt{-1}}q_1, q_2)$  and  $(q_1, q_2) \mapsto (q_1, e^{2\pi\sqrt{-1}}q_2)$  are given by  $\mathcal{O}(-E_1) \otimes (-)$  and  $\mathcal{O}(-E_2) \otimes (-)$  respectively. A direct calculation shows

$$T_1 = \varphi_{e, e_1}, \quad (6.26)$$

$$T_2 = \varphi_{e, e_2}, \quad (6.27)$$

where  $\varphi_{e, \bullet}: N \hookrightarrow O(\mathcal{N}(\check{Y}))$  is the embedding in (2.29).

## 6.5 Proof of Theorem 1.1

Let  $\Sigma$  be the fan in  $N_{\mathbb{R}}$  whose one-dimensional cones are generated by the orbit of  $e_1$  and  $e_2$  under the action of  $O^+(N)$  shown in Figure 6.5. The associated toric variety  $X_\Sigma$  shown in Figure 6.6 has a natural action of  $O^+(N)$ , and the toroidal compactification is obtained

by replacing a neighborhood of the cusp with the neighborhood of the origin in  $X_\Sigma$ . The quotient  $X_\Sigma/O^+(N)$  is obtained by first taking quotient by the infinite cyclic subgroup  $C_1 \triangleleft O^+(N)$  generated by  $g_1$ , and then by the cyclic group  $C_2 = O^+(N)/C_1$  of order two generated by  $[g_2]$ . The action of  $g_1$  on  $\Sigma$  is a ‘translation’ sending a one-dimensional cone to the one which is next next to it. The quotient of  $X_\Sigma$  by this action gives a configuration of a  $(-1)$ -curve and a  $(-5)$ -curve shown in Figure 6.7. The quotient group  $C_2$  acts on  $X_\Sigma/C_1$  by flipping along the horizontal line in Figure 6.8, and one obtains a chain of  $\mathbb{P}(1,2)$  shown in Figure 6.9 as the quotient. The normal bundles of these curves are  $\mathcal{O}_{\mathbb{P}(1,2)}(-1)$  and  $\mathcal{O}_{\mathbb{P}(1,2)}(-5)$ , which are precisely the ones that one obtains by performing iterated weighted blow-up of weight  $(1,2)$ . By contracting these curves, one obtains the cusp, which is the smooth point  $[1 : 0 : 0]$  in  $\mathbb{P}(1,3,5)$ .

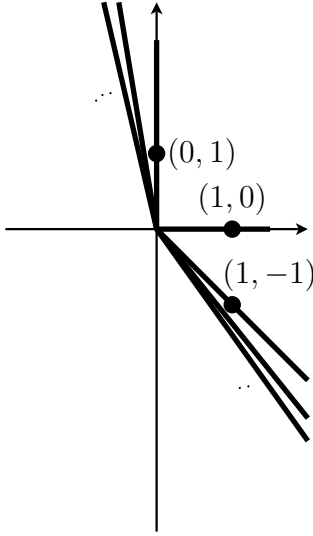


Figure 6.5: The fan  $\Sigma$  in  $N_{\mathbb{R}}$

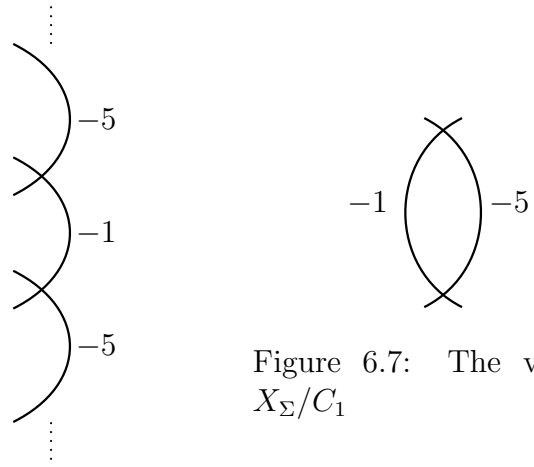


Figure 6.6: The variety  $X_\Sigma$

Figure 6.7: The variety  $X_\Sigma/C_1$

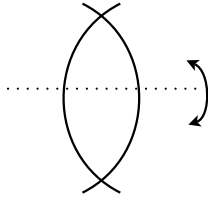


Figure 6.8: The action of  $C_2$

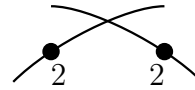


Figure 6.9: The variety  $X_\Sigma/O^+(N)$

## 7 The family associated with $A_1$

### 7.1 The secondary fan

Let  $\Delta_1$  be the reflexive polytope with 5 vertices in Figure 7.1;

$$\beta_1 = (v_1 \ v_2 \ v_3 \ v_4 \ v_5) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}. \quad (7.1)$$



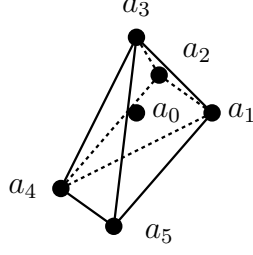


Figure 7.1: The polytope  $P_1$

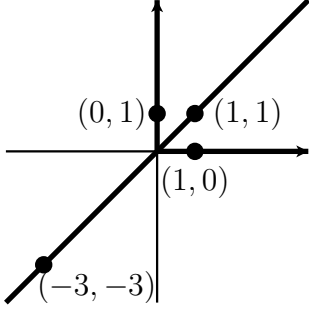


Figure 7.2: The secondary fan  $\mathcal{F}(A_1)$

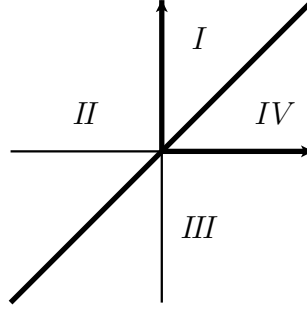


Figure 7.3: Maximal cones of  $\mathcal{F}(A_1)$

The homomorphism  $\mathbb{Z}^{n+r} \rightarrow \mathbb{L}^\vee$  in the divisor sequence (4.3) is represented by the matrix

$$\begin{pmatrix} -3 & 1 & 0 & 1 & 1 & 0 \\ -3 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The secondary fan  $\mathcal{F}(A_1)$  is given in Figure 7.2, and the secondary stack  $X_{\mathcal{F}(A_1)}$  is  $\mathbb{P}(1, 3, 3)$  blown-up at one point. Figure 7.3 shows maximal cones of the secondary fan  $\mathcal{F}(A_1)$ . The cone  $I$  corresponds to the triangulation

$$\begin{aligned} & \{a_0, a_2, a_3, a_4\} \cup \{a_0, a_1, a_2, a_4\} \cup \{a_0, a_1, a_2, a_3\} \\ & \cup \{a_0, a_3, a_4, a_5\} \cup \{a_0, a_1, a_4, a_5\} \cup \{a_0, a_1, a_3, a_5\}, \end{aligned} \quad (7.2)$$

the cone  $II$  corresponds to the triangulation

$$\{a_1, a_2, a_3, a_4\} \cup \{a_1, a_3, a_4, a_5\}, \quad (7.3)$$

the cone  $III$  corresponds to the triangulation

$$\{a_1, a_2, a_3, a_5\} \cup \{a_2, a_3, a_4, a_5\}, \quad (7.4)$$

and the cone  $IV$  corresponds to the triangulation

$$\begin{aligned} & \{a_0, a_1, a_3, a_5\} \cup \{a_0, a_1, a_2, a_5\} \cup \{a_0, a_1, a_2, a_3\} \\ & \cup \{a_0, a_3, a_4, a_5\} \cup \{a_0, a_2, a_4, a_5\} \cup \{a_0, a_2, a_3, a_4\}. \end{aligned} \quad (7.5)$$

## 7.2 The period domain

By [Nag12, Theorem 3.1], the transcendental lattice  $T_1$  of a very general member of the family  $\mathfrak{Y}_{A_1} \rightarrow X_{\mathcal{F}(A_1)}$  is isometric to  $U \perp U(3)$ . The period domain for this family is

described as follows: Let  $R = M_2(\mathbb{Z}) \cong \mathbb{Z}^4$  be a free abelian group of rank 4, equipped with the symmetric bilinear form

$$\langle v, w \rangle = -\det(v + w) + \det(v) + \det(w). \quad (7.6)$$

Then  $R$  is isometric to  $U \perp U$ , and the corresponding domain is given by

$$\mathcal{D}^+ = \left\{ Z = \begin{pmatrix} \tau_1 \tau_2 & \tau_1 \\ \tau_2 & 1 \end{pmatrix} \mid \tau_i \in \mathbb{H} \right\} \cong \mathbb{H} \times \mathbb{H}. \quad (7.7)$$

Indeed, we have

$$\langle Z, Z \rangle = 0, \quad \langle Z, \overline{Z} \rangle = -(\tau_1 - \overline{\tau}_1)(\tau_2 - \overline{\tau}_2) > 0. \quad (7.8)$$

The group  $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$  acts on  $R$  by

$$(A, B) \cdot v = AvB^T, \quad (A, B) \in SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}). \quad (7.9)$$

This action induces an action on  $\mathcal{D}$  defined by

$$Z' = (A, B) \cdot Z, \quad AZB^T \sim Z', \quad (A, B) \in SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}), \quad (7.10)$$

where  $Z \sim Z'$  if and only if  $Z = \lambda Z'$  for some  $\lambda \in \mathbb{C}^\times$ . For  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ , we have

$$AZ = \begin{pmatrix} (a\tau_1 + b)\tau_2 & a\tau_1 + b \\ (c\tau_1 + d)\tau_2 & c\tau_1 + d \end{pmatrix} \sim \begin{pmatrix} (A \cdot \tau_1)\tau_2 & A \cdot \tau_1 \\ \tau_2 & 1 \end{pmatrix}. \quad (7.11)$$

Similarly, for  $B \in SL_2(\mathbb{Z})$ , we have

$$ZB^T \sim \begin{pmatrix} \tau_1(B \cdot \tau_2) & \tau_1 \\ B \cdot \tau_2 & 1 \end{pmatrix}. \quad (7.12)$$

Hence the identification  $Z = (\tau_1, \tau_2)$  is compatible with the actions of  $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ ;

$$(A, B) \cdot Z = (A \cdot \tau_1, B \cdot \tau_2). \quad (7.13)$$

We define an involution  $\rho$  of  $R$  by  $\rho(v) = v^T$  and extend it linearly. Then we have  $(A, B)\rho = \rho(B, A)$  and the induced action of  $\rho$  on  $\mathcal{D}$  is given by  $\rho \cdot (\tau_1, \tau_2) = (\tau_2, \tau_1)$ .

Now consider the sublattice  $L$  of  $R$  defined by

$$L = \left\{ v = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in R \mid \langle v, u \rangle \equiv 0 \pmod{3} \right\}, \quad (7.14)$$

where  $u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Then we have  $L \cong U \perp U(3)$ . The subgroup of  $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$  which preserves  $L$  as a set is given by  $\Gamma' = \Gamma_0(3) \times \Gamma_0(3)$ . One has

$$O(L) = \langle \Gamma', \sigma \rangle \rtimes \langle \rho \rangle, \quad (7.15)$$

where  $\sigma \in O(L)$  is defined by

$$\sigma = (S, S): \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto S \begin{pmatrix} x & y \\ z & w \end{pmatrix} S^T = \begin{pmatrix} w/3 & -z \\ -y & 3x \end{pmatrix}, \quad S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \\ -3 & 1 \end{pmatrix}. \quad (7.16)$$

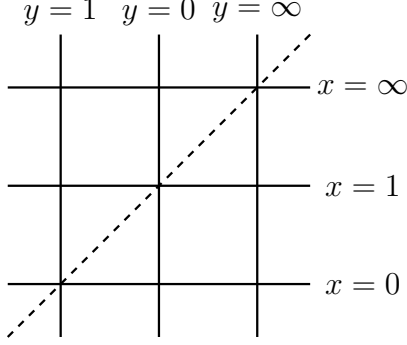


Figure 7.4:  $X_0(3) \times X_0(3)$

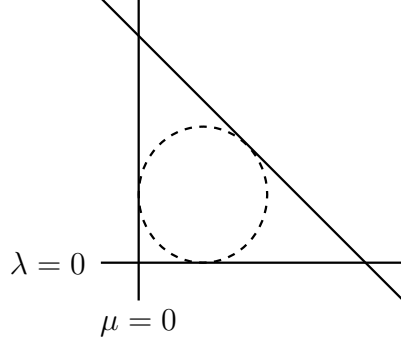


Figure 7.5:  $\overline{\mathcal{M}}_1$

The discrete group  $\Gamma^+$  is given by

$$\Gamma^+ \cong \text{Ker} \left( O^+(L) \rightarrow \text{Aut}(L^\vee/L) \right) \cong \Gamma' \ltimes \langle \rho \rangle, \quad (7.17)$$

and the moduli space is given by

$$\mathcal{M} = \mathcal{D}^+ / \Gamma^+ = X'_0(3) \times X'_0(3) / C_2, \quad (7.18)$$

where  $C_2 = \langle \rho \rangle = \mathbb{Z}/2\mathbb{Z}$  acts on the product of two copies of the modular curve  $X'_0(3) = \mathbb{H}/\Gamma_0(3)$  by permutation. Recall that the modular curve  $X'_0(3) = \mathbb{H}/\Gamma_0(3)$  has two cusps and one orbifold point of order 3. It can be compactified to  $X_0(3) \cong \mathbb{P}(1, 3, 3)$ , where the position of the cusps and the orbifold point can be set to 0, 1, and  $\infty$ . Hence the Baily-Borel-Satake compactification  $\overline{\mathcal{M}}$  of  $\mathcal{M}$  is obtained from  $\mathbb{P}(1, 3, 3)$  by the root construction along the image of the diagonal. Figure 7.4 shows the product  $X_0(3) \times X_0(3)$ , and Figure 7.5 shows the quotient  $\overline{\mathcal{M}}_1 = X_0(3) \times X_0(3) / \langle \rho \rangle$ . The dotted line in Figure 7.4 is the diagonal, which goes to the dotted line in Figure 7.5, where it has a generic stabilizer of order two.

### 7.3 The period map

The space of Laurent polynomials is given by

$$\mathbb{C}^{A_1} = \left\{ a_0 + a_1x + a_2y + a_3z + \frac{a_4}{xz} + \frac{a_5}{yz} \right\}, \quad (7.19)$$

and the discriminant is given by

$$\Delta_1 = a_0^6 + 54a_1a_3a_4a_0^3 + 54a_2a_3a_5a_0^3 + 729a_1^2a_3^2a_4^2 + 729a_2^2a_3^2a_5^2 - 1458a_1a_2a_3^2a_4a_5.$$

The dense torus  $\mathbb{L}_{\mathbb{C}^\times}^\vee$  of the secondary stack  $X_{\mathcal{F}(A_1)}$  can be written as  $\text{Spec } \mathbb{C}[\lambda^{\pm 1}, \mu^{\pm 1}]$  where

$$\lambda = \frac{a_1a_3a_4}{a_0^3}, \quad (7.20)$$

$$\mu = \frac{a_2a_3a_5}{a_0^3}. \quad (7.21)$$

The period which is holomorphic around  $\lambda = \mu = 0$  is given by

$$\eta_1(\lambda, \mu) = \sum_{n,m=0}^{\infty} (-1)^{m+n} \frac{(3n+3m)!}{(n!)^2(m!)^2(n+m)!} \lambda^n \mu^m. \quad (7.22)$$

Recall the classical relation (cf. e.g. [Vid09, (8)])

$$F_4(a, b, c, a+b-c+1; x(1-y), y(1-x)) = {}_2F_1(a, b, c; x) {}_2F_1(a, b, a+b-c+1; x) \quad (7.23)$$

between Appell's function

$$F_4(a, b, c_1, c_2; z, w) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_n(c_2)_m n! m!} z^n w^m \quad (7.24)$$

and Gauss hypergeometric function

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n, \quad (7.25)$$

where  $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$  is the Pochhammer symbol. An elementary manipulation shows

$$\eta_1(\lambda, \mu) = F_4\left(\frac{1}{3}, \frac{2}{3}, 1, 1; -27\lambda, -27\mu\right), \quad (7.26)$$

which is equal to

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; x\right) {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; x\right) \quad (7.27)$$

by (7.23) where  $-27\lambda = x(1-y)$  and  $-27\mu = y(1-x)$ . Gauss hypergeometric function

$$f_1(x) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; x\right) \quad (7.28)$$

is a solution to Gauss hypergeometric differential equation

$${}_2E_1\left(\frac{1}{3}, \frac{2}{3}, 1\right) : x(1-x) \frac{d^2 u}{dx^2} + (1-2x) \frac{du}{dx} - \frac{2}{9} u = 0. \quad (7.29)$$

This differential equation has regular singularity at  $x = 0, 1, \infty$ . By choosing a suitable solution  $f_2(x)$  to (7.29) which is holomorphic in a neighborhood of  $[0+\varepsilon, 1-\varepsilon] \subset \mathbb{R}$  in  $\mathbb{C}$  for sufficiently small  $\varepsilon$ , one obtains a map

$$x \mapsto \frac{f_2(x)}{f_1(x)} = s(x) \in \mathbb{H} \quad (7.30)$$

which is defined on  $(0, 1) \subset \mathbb{R}$  and satisfies  $s(0) = \sqrt{-1}\infty$ ,  $s(1) = 0$ ,  $s(\infty) = \frac{1}{2} + \frac{\sqrt{3}}{6}\sqrt{-1}$ . This map can be extended to a multivalued function from  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$  to  $\mathbb{H}$ . This multivalued map  $s$  sends  $\mathbb{H} \subset \mathbb{P}^1$  to a hyperbolic triangle with angles 0, 0, and  $\frac{1}{3}\pi$  in  $\mathbb{H}$ . The

monodromy of  $s$  along the closed paths  $\gamma_0$ ,  $\gamma_1$  and  $\gamma_\infty$  going around  $x = 0$ ,  $1$  and  $\infty$  are given by

$$\begin{cases} (\gamma_0)_*(s) = s + 1, \\ (\gamma_1)_*(s) = \frac{s}{-3s + 1}, \\ (\gamma_\infty)_*(s) = \frac{s - 1}{3s - 2}. \end{cases} \quad (7.31)$$

The inverse map  $\mathbb{H} \rightarrow \mathbb{P}^1$  sending  $s$  to  $x$  is a modular function with respect to  $\Gamma_0(3)$ .

The period for the family  $\mathfrak{Y}_1$  has the form

$$(\lambda, \mu) \mapsto \eta = [\eta_{11} : \eta_{12} : \eta_{13} : \eta_{14}] \in \mathcal{D}_1 = \{\eta \in \mathbb{P}(T_1) \mid (\eta, \eta) = 0, (\eta, \bar{\eta}) > 0\} \quad (7.32)$$

where  $T_1 = U \oplus U(3)$ . The connected component  $\mathcal{D}_1^+ \subset \mathcal{D}_1$  can be identified with  $\mathbb{H} \times \mathbb{H}$  by

$$(z_1, z_2) \mapsto [\eta_1 : \eta_2 : \eta_3 : \eta_4] = [3z_1z_2 : -1 : z_1 : z_2]. \quad (7.33)$$

The Gauss-Manin system for  $\mathfrak{Y}_1$  is Appell's hypergeometric differential equation of rank 4, and the periods are given by

$$\eta_{11} = 3f_2(x)f_2(y), \quad \eta_{12} = -f_1(x)f_1(y), \quad \eta_{13} = f_2(x)f_1(y), \quad \eta_{14} = f_1(x)f_2(y)$$

This period map extends to the  $xy$ -plane, which is a double cover of the  $\lambda\mu$  plane;

$$(x, y) \mapsto [\eta_{11} : \eta_{12} : \eta_{13} : \eta_{14}] = [3s(x)s(y) : -1 : s(x) : s(y)]. \quad (7.34)$$

The inverse map

$$\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad (s_1, s_2) \mapsto (x(s_1), y(s_2)) \quad (7.35)$$

is given by

$$\begin{cases} \lambda(s_1, s_2) = \frac{x(s_1)(y(s_2) - 1)}{27}, \\ \mu(s_1, s_2) = \frac{y(s_1)(x(s_2) - 1)}{27}. \end{cases}$$

## 7.4 Mirror symmetry and monodromy

The polar dual polytope is given by

$$\check{\Delta}_1 = \text{Conv} \{(2, 2, -1), (2, -1, -1), (-1, -1, -1), (-1, 2, -1), (-1, -1, 2)\}.$$

The ambient space  $\check{X}$  for the mirror family  $\check{\mathcal{Y}}$  is the toric weak Fano 3-fold of Picard number 2, which is obtained as a crepant resolution of a toric Fano 3-fold of Picard number 1 with an ordinary double point.

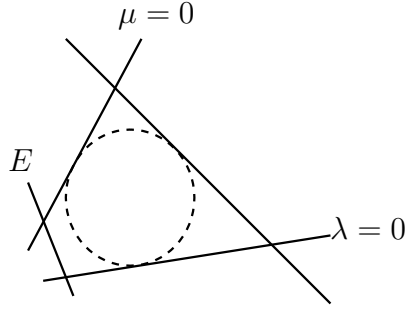


Figure 7.6:  $X_{\mathcal{F}(A_1)}$

The Picard lattice of a very general member of  $\check{\mathcal{Y}}$  is generated by the restrictions  $E_1$  and  $E_2$  of the toric divisors  $D_4$  and  $D_5$  of the ambient space, and one has

$$M_{\Delta_1} = \text{Pic } Y_1 = E_8 \perp E_8 \perp U(3), \quad (7.36)$$

$$M_{\Delta_1}^\perp = U \perp U(3), \quad (7.37)$$

$$M_{\check{\Delta}_1} = \text{Pic } \check{Y}_1 = U(3), \quad (7.38)$$

$$M_{\check{\Delta}_1}^\perp = U \perp E_8 \perp E_8 \perp U(3), \quad (7.39)$$

so that Conjecture 3.1 holds in this case.

The numerical Grothendieck group  $\mathcal{N}(\check{Y})$  is isometric to  $U \perp \text{Pic}(\check{Y})$ , and the nef cone of  $\check{X}$  is generated by  $D_4 + D_5$  and  $D_5$ . One can show that the monodromies around  $q_1 = \lambda$  and  $q_2 = \lambda^{-1}\mu$  are given by  $\varphi_{e, e_1+e_2}$  and  $\varphi_{e, e_2}$  just as in the case of  $A_0$ . For the other crepant resolution, the nef cone is generated by  $D_4$  and  $D_4 + D_5$ , and the monodromies are given by  $\varphi_{e, e_1}$  and  $\varphi_{e, e_1+e_2}$ . As a result, the toroidal compactification is given by the fan in  $N_{\mathbb{R}}$  whose one-dimensional cones are spanned by  $e_1$ ,  $e_1 + e_2$ , and  $e_2$ . This blows up the intersection point of two components of the cusp as shown in Figure 7.6, and the resulting stack is precisely the stack  $\check{X}_{\mathcal{F}(A_1)}$  obtained from  $X_{\mathcal{F}(A_1)}$  by the root construction along the strict transform of the diagonal in  $X_0(3) \times X_0(3)$ .

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